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Axisymmetric turbulent boundary layer along a circular cylinder at constant pressure

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A constant-pressure axisymmetric turbulent boundary layer along a circular cylinder of radius a is studied at large values of the frictional Reynolds number a_+ (based upon a) with the boundary-layer thickness δ of order a. Using the equations of mean motion and the method of matched asymptotic expansions, it is shown that the flow can be described by the same two limit processes (inner and outer) as are used in two-dimensional flow. The condition that the two expansions match requires the existence, at the lowest order, of a log region in the usual two-dimensional co-ordinates (u_+, y_+). Examination of available experimental data shows that substantial log regions do in fact exist but that the intercept is possibly not a universal constant. Similarly, the solution in the outer layer leads to a defect law of the same form as in two-dimensional flow; experiment shows that the intercept in the defect law depends on δ/a. It is concluded that, except in those extreme situations where a_+ is small (in which case the boundary layer may not anyway be in a fully developed turbulent state), the simplest analysis of axisymmetric flow will be to use the two-dimensional laws with parameters that now depend on a_+ or δ/a as appropriate.

1. Introduction

The study of the effects of transverse curvature on turbulent boundary layers is of interest in numerous engineering applications. Some examples are axial flow along slender bodies of revolution, ship models and other three-dimensional objects. The effects of transverse curvature may be expected to become significant when the thickness δ of the boundary layer is comparable to or larger than a typical transverse dimension (say a) of the body.

The simplest problem of this class, where transverse curvature effects are important, is that of the axisymmetric turbulent boundary layer along a long circular cylinder with zero pressure gradient. This problem has been extensively studied in the literature (Richmond 1957; Yu 1958; Yasuhara 1959; Rao 1967*a*; Chin, Hulesbos & Hunnicutt 1967; Willmarth & Yang 1970; Rao & Keshavan 1972), but there is no agreement yet regarding the appropriate similarity laws for

Author(s)	Proposed law of wall in log region	Parameter
Classical two-dimensional law	$U_+ = A_1 \ln y_+ + A_2$	A_1, A_2 universal
Richmond (1957)	$U_+ = A_1 \ln [y_+(1+y_+/2a_+)] + A_2$	A_1, A_2 same as in two-dimensional law
Yu (1958)	$U_+ = A_1 \ln y_+ + A_2$	A_1 universal and $A_2 = A_2(R_a)$
Chin *et al.* (1967)	$U_+ = A_1 \ln y_+ + A_2$	A_1 and A_2 functions of R_a
Rao & Keshavan (1972)	$U_+ = A_1 \ln [a_+ \ln (1+y_+/a_+)] + A_2$	$A_1 = A_1(R_a)$ and $A_2 = A_2(a_+, R_a)$

TABLE 1. Different proposals for velocity distribution in log region of the wall layer

the mean flow. Particular attention has been given by all workers to the wall law in the overlap or 'log' region; the different proposals made for the velocity distribution in this region are shown in table 1.

With the exception of Yu (1958) and Chin *et al.* (1967), all authors have assumed that the classical two-dimensional law of the wall is in need of modification, which they propose to achieve by altering the argument of the similarity function (but not the function itself). For example, Richmond (under Coles's guidance) has proposed that, if the two-dimensional wall law can be written as

$$U_+ = F_1(y_+) \quad (U_+ = U/U_\tau, \quad y_+ = yU_\tau/\nu) \tag{1}$$

(where U_τ is the friction velocity and U_+ and y_+ the usual wall variables), the appropriate form in axisymmetric flow involves only a change in the argument of the function F_1:

$$U_+ = F_1[y_+(1+y_+/2a_+)], \tag{2}$$

where $a_+ = aU_\tau/\nu$ is the frictional Reynolds number. Richmond's own experiments appeared to support (2); later Yasuhara (1959) and Willmarth & Yang (1970) also presented their measurements in the above form with results quite similar to those of Richmond.

Rao (1967*a*) has proposed that the axisymmetric wall variable is the one that will preserve the (linear) form of the law of the wall in the viscous sublayer. This suggests that in axisymmetric flow

$$U_+ = F_1[a_+ \ln(1+y_+/a_+)]. \tag{3}$$

The experiments of Rao & Keshavan (1972) showed that F_1 did possess a log region where (3) could be written as

$$U_+ = A_1 \ln[a_+ \ln(1+y_+/a_+)] + A_2,$$

but the quantities A_1 and A_2 could not be considered as universal constants; in fact they found that $A_1 = A_1(R_a)$ and $A_2 = A_2(a_+, R_a)$, where $R_a = U_\infty a/\nu$ is the Reynolds number based on the free-stream velocity U_∞.

Coles (1970) has pointed out that Richmond's co-ordinates reflect the effects of transverse curvature on the continuity equation, given Coles's streamline hypothesis; while the co-ordinate of Rao reflects the effects of transverse curvature on the momentum equation.

Less attention has been paid to describing the flow in the outer region of axisymmetric turbulent boundary layers. Rao & Keshavan (1972) have found that for a given R_a the outer-layer velocity defect $(U_\infty - U)/U_\tau$ yields similarity in terms of the variable r_+, based on the radius $r = a+y$. However, their wall and defect laws cannot be matched and as a result a skin-friction law does not follow. Thus Rao's law is not complete. On the other hand Yu (1958) and Chin *et al.* (1967) have presented their measurements in classical two-dimensional defect-law co-ordinates

$$\begin{aligned}(U_\infty - U)/U_\tau &= G(y/\delta) \\ &\sim A_3 \ln(y/\delta) + A_4, \quad y/\delta \ll 1.\end{aligned}$$

Yu (1958) found that A_3 is a universal constant and $A_4 = A_4(R_a)$ while Chin *et al.* (1967) found that neither A_3 nor A_4 is a universal constant.

When compared with two-dimensional flow, the properties of the axisymmetric turbulent boundary layer with zero pressure gradient may be expected to depend on two additional non-dimensional parameters, namely a_+ and δ/a. Values of these parameters covered in the various experiments reported in the literature are shown in table 2. It may be seen that, except in some of Richmond's experiments, a_+ is larger than 20 and δ/a is between 0·1 and 12. It therefore appears that for the bulk of these flows it should be useful to consider the limit $a_+ \gg 1$ and δ/a of order unity. The main aim of the present work is to formulate an asymptotic theory valid under such conditions.

Before making a detailed analysis, we note here one immediate consequence of the above limit. It is well known that, in general, the total stress τ_t (= viscous + Reynolds) does not remain constant in the inner region of an axisymmetric turbulent layer, but that the stress moment $r\tau_t$ does. Thus

$$r\tau_t = a\tau_w \quad \text{or} \quad (1 + y_+/a_+)\tau_t = \tau_w. \tag{4}$$

For large values of a_+ the expression (4) can be expanded as

$$\tau_t = \tau_w(1 - y_+/a_+ \ldots). \tag{5}$$

In the sublayer, where the viscous stress is dominant, the velocity profile is easily found by integration to be

$$\begin{aligned}U_+ &= a_+ \ln(1 + y_+/a_+) \\ &= y_+ - y_+^2/a_+ \ldots \quad \text{for} \quad a_+ \gg 1.\end{aligned} \tag{6}$$

The corresponding sublayer expression for constant-pressure two-dimensional boundary layers is (Rotta 1962, p. 59)

$$U_+ = y_+ + O(y_+^4).$$

Thus, if a_+ is large, to order $1/a_+$ the total stress in an axisymmetric wall layer can still be taken as constant, and consequently the sublayer profile as linear. In other words, for large values of a_+ the effects of transverse curvature on the inner layer

Author	R_a	x/a	a_+	δ/a
Richmond (1957)	40200	16	1615	1·18
	40200	18	1580	1·52
	40200	20	1525	1·84
	253	1600	12·5	46
	253	1600	12·8	72
	94	1600	5	64
Yu (1959)	15250	96	751	1·6
	30740	96	1150	1·5
	45060	84	1630	1·4
Willmarth & Yang (1970)	134000	192	4710	2
	134000	256	4635	2·15
	70200	192	2655	2·56
	70200	256	2595	2·84
Rao & Keshavan (1972)	218500	5·3	10500	0·193
		7·7	9800	0·215
		12	9000	0·31
		14·2	8700	0·36
		16·4	8500	0·345
	106000	5·3	5350	0·227
		7·7	5000	0·237
		12	4610	0·365
		14·2	4510	0·436
		16·4	4380	0·397
	3940	24	262	2·08
		68	212	2·36
		92	196·5	3·02
		124	180	3·18
		160	174	4·13
	1620	48	127·8	4
		136	93·5	4·65
		184	92	5·51
		248	84	6·11
		320	79·5	6·58
	1420	24	104·5	2·63
		68	76	2·82
		92	70	3·1
		124	62	3·57
		160	51·8	3·72
	1320	48	102·4	5
		136	78·9	5·3
		184	71·9	6·3
		248	66·5	6·26
		320	62·2	6·9
	825	96	52·8	10·82
		272	34·1	10·75
		365	28·2	11·45
		499	25·1	10·9
		640	22·5	11·56
	425	96	38·4	8·15
		272	29·2	11·30
		365	27·2	9·75
		499	26·6	10·65
		640	23·1	11·6

TABLE 2. Values of a_+ and δ/a covered in various experiments

must be capable of being regarded as higher-order perturbations to a basically two-dimensional flow. A consequence of this argument is that the measurements quoted above must show a logarithmic region when plotted in two-dimensional co-ordinates. We shall show that this is indeed the case in spite of the fact that the validity of the two-dimensional wall and defect laws in axisymmetric flow has been widely disputed.

2. Analysis

Consider steady axially symmetric turbulent flow of an incompressible fluid along the convex surface of a circular cylinder with zero pressure gradient. The boundary-layer equations for the mean motion are

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{V}{a+y}=0, \tag{7}$$

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{1}{a+y}\frac{\partial}{\partial y}\left[(a+y)\left\{\nu\frac{\partial U}{\partial y}+\tau\right\}\right]. \tag{8}$$

Here x is the streamwise co-ordinate, y the normal co-ordinate measured from the surface and U and V are the mean velocity components in the x and y directions respectively. The boundary conditions are

$$U=V=\tau=0 \quad \text{at} \quad y=0, \tag{9a}$$

$$U\rightarrow U_\infty,\ \tau\rightarrow 0 \quad \text{as} \quad y\rightarrow\infty. \tag{9b}$$

The momentum integral obtained from above set of equations is

$$\frac{d\theta_1}{dx}=\tfrac{1}{2}c_f=\frac{U_\tau^2}{U_\infty^2},\quad \theta_1=\int_0^\delta\frac{U}{U_\infty}\left(1-\frac{U}{U_\infty}\right)\left(1+\frac{y}{a}\right)dy. \tag{10a}$$

The momentum thickness θ is related to momentum length θ_1 by

$$\theta_1=\theta(1+\theta/2a). \tag{10b}$$

We now study the two limits (inner and outer) and the two corresponding asymptotic expansions for large values of the Reynolds number a_+.

2.1. *Outer layer*

It does not seem to have been realized in earlier work on the asymptotics of turbulent shear flows that an explicit estimate of flow variations in the streamwise direction is both necessary and useful. If L is the scale of these variations, we may infer from the momentum integral (10) that $L=O(\theta_1 U_\infty^2/U_\tau^2)$. Defining

$$X=\int_0^x\frac{dx}{L},\quad Y=y/\delta,\quad L=\theta_1 U_\infty^2/U_\tau^2, \tag{11a–c}$$

$$U=U_\infty\hat{U},\quad V=(\delta/L)U_\infty\hat{V},\quad \tau=U_\tau^2 T, \tag{12a–c}$$

and introducing these variables into the equations of motion (7) and (8), we get

$$\frac{\partial \hat{U}}{\partial X} - Y\frac{\delta_X}{\delta}\frac{\partial \hat{U}}{\partial Y} + \frac{\partial \hat{V}}{\partial Y} + \frac{\delta}{a+\delta Y}\hat{V} = 0, \tag{13a}$$

$$\hat{U}\frac{\partial \hat{U}}{\partial X} + \left(\hat{V} - Y\frac{\delta_X}{\delta}\hat{U}\right)\frac{\partial \hat{U}}{\partial Y} = \epsilon\left[\frac{1}{a+\delta Y}\frac{\partial}{\partial Y}(a+Y\delta)T + \frac{U_\infty}{\delta_+ U_\tau}\frac{1}{a+\delta Y}\frac{\partial}{\partial Y}(a+Y\delta)\frac{\partial \hat{U}}{\partial Y}\right]. \tag{13b}$$

Here ϵ is a parameter representing the momentum defect, defined by

$$\epsilon = \theta_1/\delta. \tag{13c}$$

For large values of a_+ the momentum defect may be expected to be much less than unity. Equations (13a, b) then show that the velocity profile at any point in the outer region may be written in functional form as

$$\begin{aligned} u_- &= (U_\infty - U)/\epsilon \\ &= u_-(X, Y, \delta/a, \epsilon_X/\epsilon, \epsilon U_\infty/\delta_+ U_\tau). \end{aligned} \tag{14}$$

As $a_+ \to \infty$ and $\delta/a = O(1)$ we consider the asymptotic expansions

$$\left.\begin{aligned} \hat{U} &= 1+\epsilon[U_1(X,Y;\delta/a)+E(X,R)\,U_2(X,Y;\delta/a)+\ldots], \\ \hat{V} &= \epsilon[V_1(X,Y;\delta/a)+E(X,R)V_2(X,Y;\delta/a)+\ldots], \\ T &= T_1(X,Y;\delta/a)+E(X,R)\,T_2(X,Y;\delta/a)+\ldots. \end{aligned}\right\} \tag{15}$$

Here $\epsilon(X,R)$ and $E(X,R)$ are gauge functions to be determined. The lowest-order equations are therefore

$$\frac{\partial U_1}{\partial X} - Y\frac{\delta_X}{\delta}\frac{\partial U_1}{\partial Y} + \frac{\partial V_1}{\partial Y} + \frac{\delta}{a+\delta Y}V_1 + \frac{\epsilon_X}{\epsilon}U_1 = 0, \tag{16}$$

$$\frac{\partial U_1}{\partial X} - Y\frac{\delta_X}{\delta}\frac{\partial U_1}{\partial Y} = \frac{1}{a+\delta Y}\frac{\partial}{\partial Y}(a+\delta Y)\,T_1 - \frac{\epsilon_X}{\epsilon}U_1. \tag{17}$$

If (as we shall show in §2.3) the terms multiplying ϵ_X/ϵ can be omitted, these equations are identical to those for an axisymmetric wake (Townsend 1956).

2.2. *Inner layer*

We may introduce the following inner variables:†

$$\left.\begin{aligned} X &= \int_0^x \frac{dx}{L}, \quad y_+ = yU_\tau/\nu, \\ U &= U_\tau\,\partial f(X,y_+)/\partial y_+, \quad V = (\nu/L)\,g(X,y_+), \quad \tau = U_\tau^2\,\tilde{\tau}(X,y_+). \end{aligned}\right\} \tag{18}$$

Substituting these variables in the equation of continuity (7) the relation between f and g is obtained as

$$(a_+ + y_+)g = -\int_0^{y_+}\left[(a_+ + y_+)f_{Xy_+} + \frac{U_{\tau X}}{U_\tau}(y_+ f_{y_+})_{y_+}\right]dy_+. \tag{19}$$

† For streamwise variations in the inner region a length scale $\mathscr{L}$ different from L may be introduced, but the inner equation (20) shows that this length scale occurs in terms which are of very much higher order, suggesting that the streamwise variations in the inner region are very much smaller than those in the outer region.

The equation (8) for the axial momentum reduces to

$$f_{y_+y_+y_+}+\tilde{\tau}_{y_+}+\frac{1}{(a_++y_+)}(\tilde{\tau}+f_{y_+y_+})=\frac{1}{L_+}\left[\frac{U_{\tau X}}{U_\tau}\left(f_{y_+}^2+\frac{1}{a_++y_+}f_{y_+y_+}\int_0^{y_+}y_+f_{y_+}dy_+\right)\right.$$
$$\left.+f_{y_+}f_{Xy_+}-f_Xf_{y_+y_+}+\frac{1}{a_++y_+}f_{y_+y_+}\int_0^{y_+}f_X dy_+\right]. \quad (20)$$

Equation (20) suggests that the functional form for a typical variable, say f, is

$$f=f(X,y_+,a_+^{-1},L_+^{-1},U_{\tau X}/U_\tau L_+). \quad (21)$$

The inner asymptotic expansions are now written as

$$\left.\begin{aligned} f &= f_1+\Delta f_2+\ldots,\\ g &= g_1+\Delta g_2+\ldots,\\ \tilde{\tau} &= \tau_1+\Delta\tau_2+\ldots. \end{aligned}\right\} \quad (22)$$

Here Δ is a gauge function yet to be determined. The equations for the leading approximation

$$f_{1y_+y_+y_+}+\tau_{1y_+}=0 \quad (23)$$

are thus identical to those for two-dimensional flows, and show that the total stress in the wall layer remains constant.

2.3. *Matching*

According to what may be called the Millikan–Kolmogorov principle, at sufficiently large Reynolds numbers the inner and outer solutions must match; thus

$$\lim_{Y\to 0}U_\infty U_\tau^{-1}[1+\epsilon U_1(X,Y)]\sim\lim_{y_+\to\infty}f_{1y_+}(X,y_+). \quad (24)$$

As $R_a\to\infty$, the first term on the left-hand side approaches infinity and the matching requires that the right-hand side be unbounded for large y_+, say like the function $F(y_+)$. This function can be differentiated with respect to y (note that this procedure, adopted by Millikan (1938), would be invalid if $F(y_+\to\infty)$ did not diverge; see Afzal & Narasimha 1976) to get

$$\lim_{Y\to 0}\frac{\epsilon U_\infty}{U_\tau}Y\frac{\partial U_1}{\partial Y}\sim\lim_{y_+\to\infty}y_+\frac{\partial F}{\partial y_+}. \quad (25)$$

If we choose $\epsilon U_\infty/U_\tau$ to be of order unity, say

$$\epsilon=U_\tau/U_\infty, \quad (26)$$

then the matching condition (25) implies that each side of it approaches a constant independent of Y and y_+, say $1/K$, so that

$$U_1=K^{-1}\ln Y-D \quad \text{as} \quad Y\to 0, \quad (27a)$$

$$F=K^{-1}\ln y_++C \quad \text{as} \quad y_+\to\infty. \quad (27b)$$

On matching the tangential component of the velocity we get the skin-friction law

$$U_\infty/U_\tau=K^{-1}\ln\delta_++C+D. \quad (28)$$

It is interesting to note that from (11c) and (26) it follows that

$$\delta/L=U_\tau/U_\infty. \quad (29)$$

In Millikan's (1938) overlap argument the dependence of the constants C and D in (27) and (28) on the parameters of the problem remains hidden (Coles 1971). The equations for the outer layer, (16) and (17), contain the parameter δ/a and thus the coefficient D could depend upon δ/a. From a similar argument for the inner layer it follows that C is a universal constant.

To gain insight into the structure of higher-order effects in the outer and inner layers, we now compare the orders of various parameters in the full governing equations. Let us first consider the outer-layer equation (14), which shows dependence on the two parameters ϵ_X/ϵ and δ_+^{-1}. From the skin-friction law (28) we get

$$\frac{\epsilon_X}{\epsilon} = -\epsilon\left(\frac{1}{K}\frac{\delta_X}{\delta} + D_X\right) = O(\epsilon). \tag{30}$$

The ratio of the two parameters is

$$\epsilon_X/\epsilon\delta_+ \simeq O(\nu/U_\infty \delta), \tag{31}$$

which approaches zero as R_δ becomes large. Thus the parameter ϵ_X/ϵ is dominant compared with δ_+^{-1} and the gauge function E in the outer expansions (15) is taken to be

$$E = \epsilon. \tag{32}$$

The inner-layer equation (21) contains three parameters: a_+^{-1}, L_+^{-1} and $U_{\tau X}/U_\tau L_+$. From relation (30) it is obvious that of the last two L_+^{-1} is the dominant one. We now estimate the orders of magnitude of the first two parameters. The ratio of L_+^{-1} to a_+^{-1} can easily be shown to be

$$L_+^{-1}/a_+^{-1} = a\epsilon/\delta. \tag{33}$$

At large values of the Reynolds number $\epsilon \ll 1$, and thus for the present case when δ/a is of order unity a_+^{-1} is the dominant parameter compared with L_+^{-1}. Thus in the inner asymptotic expansions (22) the gauge function is chosen to be

$$\Delta = 1/a_+. \tag{34}$$

This implies that in the inner layer the effects of transverse curvature are much more important than those of inertia.

3. Results and discussion

The main results for the velocity profile are

$$\frac{U}{U_\tau} = \frac{1}{K}\ln y_+ + C + O\left(\frac{1}{a_+}\right) \quad \text{as} \quad y_+ \to \infty, \tag{35}$$

$$\frac{U_\infty - U}{U_\tau} = -\frac{1}{K}\ln Y + D\left(\frac{\delta}{a}\right) + O\left(\frac{U_\tau}{U_\infty}\right) \quad \text{as} \quad Y \to 0 \tag{36}$$

and the skin-friction law is

$$\frac{U_\infty}{U_\tau} = \frac{1}{K}\ln \delta_+ + C + D + O\left(\frac{U_\tau}{U_\infty}\right). \tag{37}$$

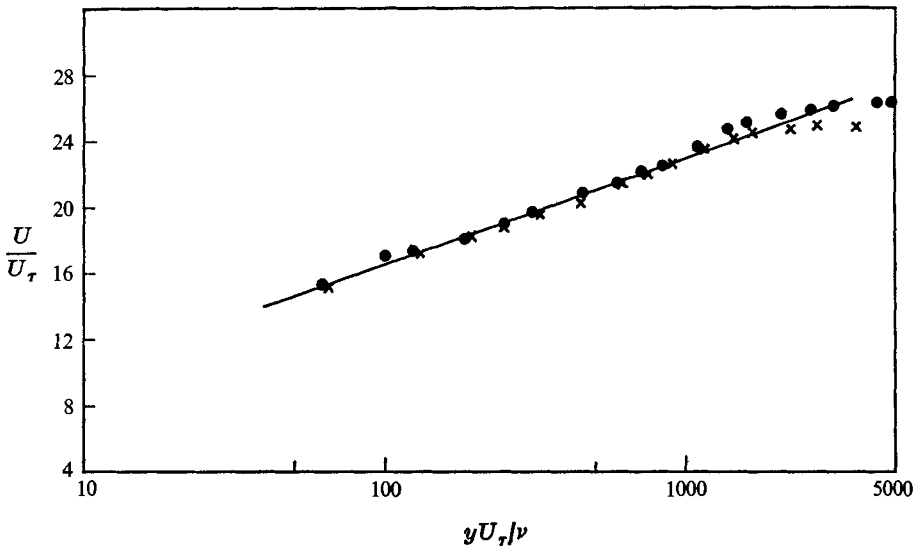


FIGURE 1. Law of the wall: Richmond's measurements.
$R_a = 40200$. ×, $x = 8$ ft; ●, $x = 10$ ft.

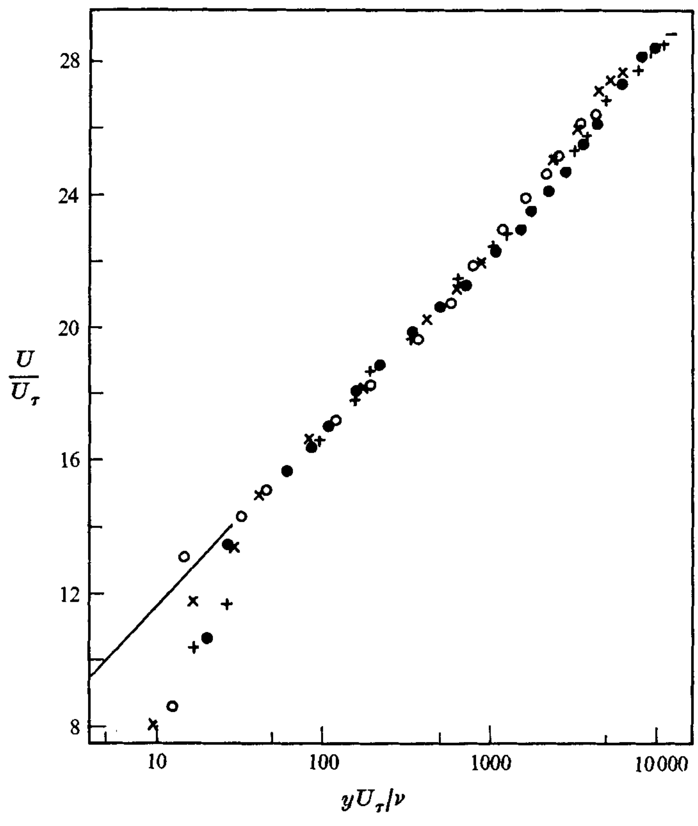
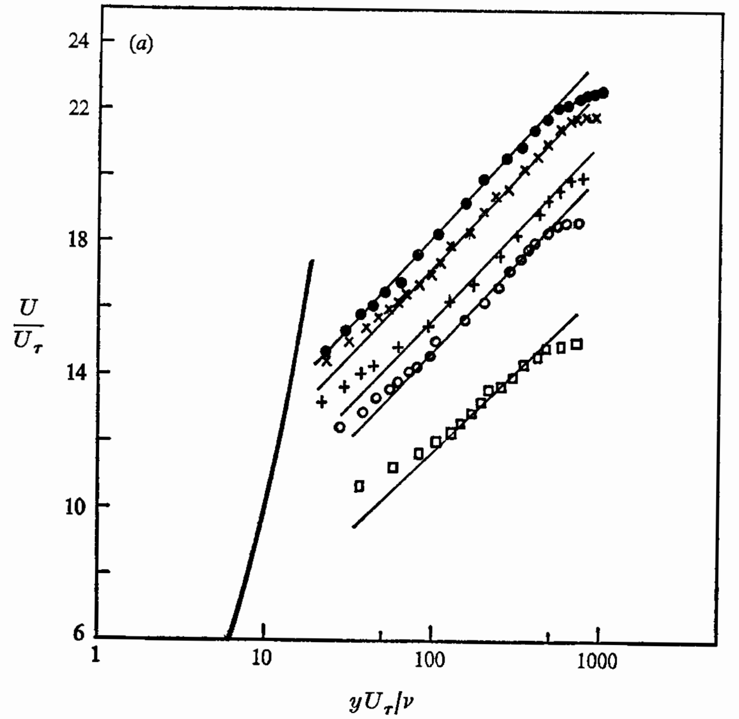


FIGURE 2. Law of the wall: Willmarth & Yang's measurements.
$R_a = 70200$: ○, $x = 24$ ft; ×, $x = 32$ ft. $R_a = 134000$: ●, $x = 24$ ft; +, $x = 32$ ft.



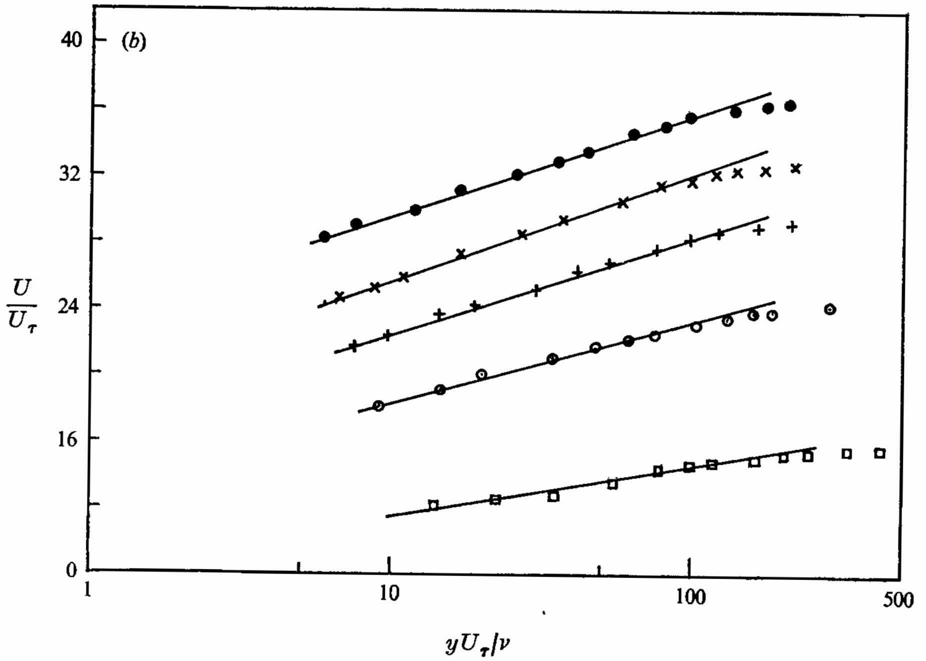


FIGURE 3. Law of the wall: Rao & Keshavan's measurements. (*a*) $R_a = 3940$. (*b*) $R_a = 825$. ●, $x = 20$ in.; ×, $x = 15{\cdot}5$ in.; +, $x = 11{\cdot}5$ in.; ○, $x = 8{\cdot}5$ in.; □, $x = 3$ in. The heavy curve in (*a*) represents the sublayer equation (6).

The work of Yu (1958) and Chin *et al.* (1967) shows that when their measurements are plotted in two-dimensional wall-law co-ordinates a logarithmic region is clearly visible. To check further the present result (35) data of Richmond (1957) and Willmarth & Yang (1970) are plotted in the wall-law co-ordinates of (35) in figures 1 and 2. The measurements of Rao & Keshavan (1972) for $Ra = 3940$ and 825 are displayed in figures 3(*a*) and (*b*) while those for other values of the Reynolds number are shown in Afzal & Narasimha (1975). All these figures also show the existence of a substantial logarithmic region. Thus all the available measurements are overwhelmingly in favour of a logarithmic law in classical two-dimensional co-ordinates.

As no reliable values of the skin friction from direct measurements are available the determination of the constants in the wall law is plagued by uncertainties in c_f. Each author has deduced the skin friction by a different indirect method. For example, Yu (1958) and Willmarth & Yang (1970) determined the wall shear from a Preston tube, using the calibrations, respectively, of Landweber & Siao (1958) and Patel (1965). Richmond obtained the wall friction by fitting data to his wall law (2). However, Rao & Keshavan (1972) found that in some of Richmond's flows the momentum thickness increased downstream, suggesting that the flow may not have been axisymmetric. Chin *et al.* (1967) and Rao & Keshavan (1972) obtained the skin friction from the momentum integral by using the measured streamwise momentum thickness. This method involves a differentiation which is not altogether reliable. For some of the above measurements Patel (1973) has deduced the wall friction from Clauser's plot (which applies strictly to the usual flat-plate wall law) on the basis of the following argument. As the departure of the axisymmetric wall flow from the two-dimensional law is gradual, being much smaller in the sublayer and mixing region than at larger y_+, a reasonable estimate of the friction can be obtained from Clauser's plot provided that due emphasis is placed on experimental points in the sublayer and mixing region. Patel has in this way determined the friction for five experiments of Rao & Keshavan and has shown that in two of them his value differs from theirs by as much as 20%. We believe that at present Patel's procedure is the most rational and will therefore adopt here the c_f values derived by him.

Before we study the dependence of the intercept on various parameters, we first make a few remarks about the data of Rao & Keshavan, particularly those taken at earlier stations. The experiments at $R_a = 218\,500$ and $106\,000$ were conducted using a 5·5 in. diameter model 6 ft long with an ogival nose-piece of slenderness ratio 3:1. The boundary layer was artificially tripped by emery paper at a distance of about a third of the cylinder's radius from the apex and by a tripping wire ahead of it; and the first and last measuring stations were at only $x/a = 5{\cdot}2$ and $16{\cdot}3$ respectively. In view of the known slow recovery of boundary layers from the effect of tripping devices (Coles 1962), it is likely that at least at the initial stations the boundary layer may not yet have achieved a natural state. In fact the boundary layers were also artificially tripped for all measurements of Rao & Keshavan in the range $425 \leqslant R_a \leqslant 3940$. In some cases (figure 3) the velocity profile shows a negative wake component on a wall-law plot. These figures show that, for a given R_a, as the flow develops in the downstream direction

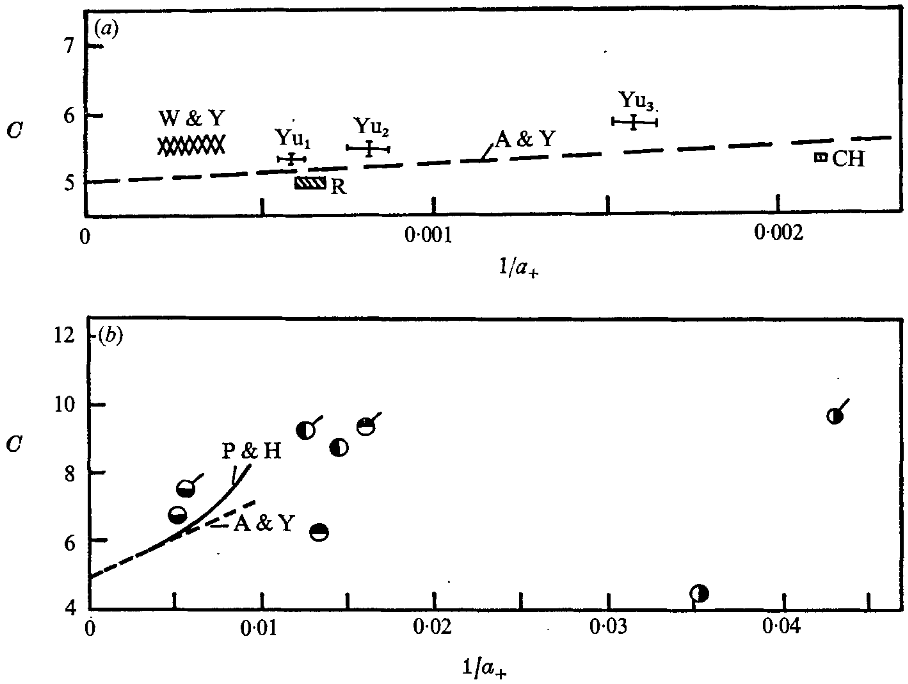


FIGURE 4. Variation of additive term in law of the wall with respect to inverse of frictional Reynolds number. (*a*) W & Y, Willmarth & Yang; R, Richmond; C, Chin *et al.*; A & Y, Afzal & Yajnik. (*b*) P & H, Patel & Head. Half-filled circles are from Rao & Keshavan's data. Circles with flags are the corresponding point when skin friction obtained by Patel is employed.

the negative wake component disappears (the profile at this location is called the marginal profile by Rao & Keshavan); a positive wake appears further downstream. Furthermore the disappearance of the negative wake at larger values of R_a is very rapid. In connexion with two-dimensional flows Coles (1962) has discussed the appearance of such marginal profiles at low values of the momentum Reynolds number. Furthermore, it has been argued by several workers (Rao & Keshavan 1972; Patel 1973) that the effects of transverse curvature are qualitatively similar to those of a favourable pressure gradient. Therefore, if the curvature effects are stronger there is a possibility of relaminarization of axisymmetric turbulent boundary layers. Different criteria have been proposed by various workers. For example Rao & Keshavan (1972) suggested that an axisymmetric turbulent boundary layer will persist if $\bar{R}_a > 15000$ and that below this value relaminarization may occur for some R_x. This value is close to the value 11 000 calculated by Rao (1967*b*) below which all small disturbances to laminar axisymmetric boundary layers will damp out. On the other hand Patel (1973) suggested that for $a_+ < 28$, for all R_x, axisymmetric boundary layers must be regarded as transitional. The measurements of fully developed pipe flow of Patel & Head (1969) show that for $R_a < 3000$ (or equivalently $a_+ < 106$) the flows are transitional. These considerations suggest that many of the low R_a experiments of Rao &

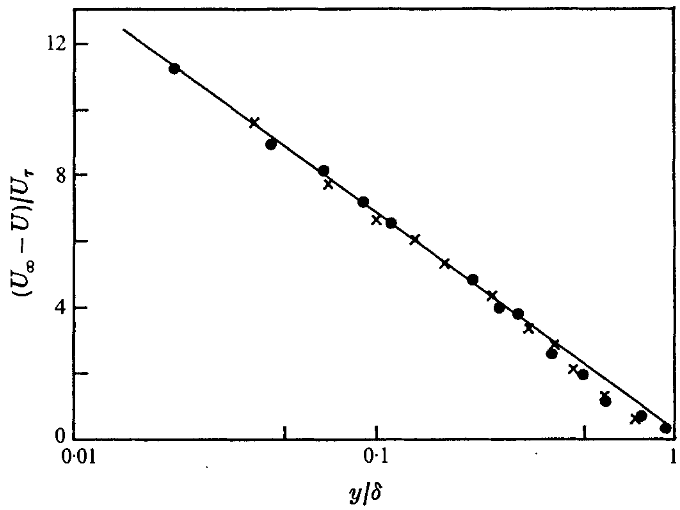


FIGURE 5. Velocity defect law: Richmond's measurements. $R_a = 40200$. ×, $x = 8$ ft; ●, $x = 10$ ft.

Keshavan (1972) and Richmond (1957) should be considered suspect as the flow may not have been in a fully turbulent state. Thus for the determination of constants in the wall law we use measurements at stations further downstream (so as to ensure that tripping effects have become small) and larger values of R_a (to ensure that the boundary layer is not transitional).

Examination of the above data shows that the intercept of the log law is possibly not a universal constant. The present theory shows that constants in the wall law (35) have to be universal if a_+ is sufficiently large. Thus any departure of additive terms in the wall law from their asymptotic values could be regarded as a higher-order effect. An inspection of the relations (21), (22) and (34) governing the inner flow suggests that the higher-order effects are of order $1/a_+$. Furthermore, the dependence of additive terms in the wall law for axisymmetric boundary layers is, in a sense, similar to that observed by Patel & Head (1969) in their measurements of fully developed pipe flow. It has been shown by Afzal & Yajnik (1973) that the appropriate wall law for such a flow is

$$U_+ = f(y_+, 1/a_+)$$

and that the intercept is a function of $1/a_+$. For small values of $1/a_+$ the intercept B in pipe flows is given by (Afzal & Yajnik 1973)

$$B = 5 + 236/a_+. \tag{38}$$

The values of the intercept B obtained from various sources are plotted against $1/a_+$ in figure 4. In the same figure the correlation (38) for pipe flow is also shown. The figure shows that the data from the various sources lie very close to the results for pipe flow. The scatter in the data is such that it is not possible to assert that B does increase with $1/a_+$, but the measurements would not be inconsistent with (38) either.

In contrast to the wall law, not much attention has been paid to describing the

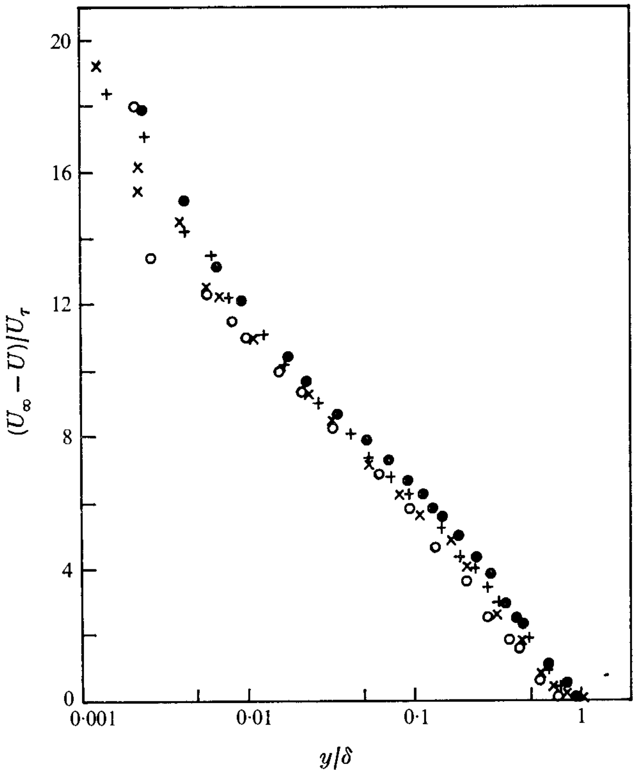


FIGURE 6. Velocity defect law: Willmarth & Yang's measurements. $R_a = 70200$: ○, $x = 24$ ft; ×, $x = 32$ ft. $R_a = 134000$: ●, $x = 24$ ft; +, $x = 32$ ft.

flow in the outer region of axisymmetric turbulent boundary layers. The measurements of Yu (1958) and Chin *et al.* (1967), when plotted in two-dimensional defect-law co-ordinates, show a logarithmic region. Rao & Keshavan (1972), on the basis of their own measurements, have however concluded that the two-dimensional defect-law co-ordinates do not show any promise. Here, in figures 5–7 we have displayed data from various sources (Richmond 1957; Willmarth & Yang 1970; Rao & Keshavan 1972) in two-dimensional defect-law co-ordinates and it is interesting to note the clear existence of a substantial logarithmic region. Thus all the available measurements are again overwhelmingly in favour of a logarithmic law in classical defect co-ordinates. It is, however, also clear from these plots that the value of the intercept D is not a universal constant. The present theory suggests that to lowest order D could be a function of δ/a. To check this, the values of the intercept from various sources mentioned above and Singh (1973) are plotted against δ/a in figure 8. The figure shows that within the accuracy of the data D correlates well with δ/a. As $\delta/a \to 0$ the value of D approaches the flat-plate value.

Thus from the above it follows that the classical form of two-dimensional wall and defect laws can describe axisymmetric turbulent boundary layers provided that $1/a_+$ and δ/a are small. For moderately small values of $1/a_+$ and for δ/a of order unity, the intercept in the logarithmic wall law depends on $1/a_+$ and that in the defect law on δ/a.

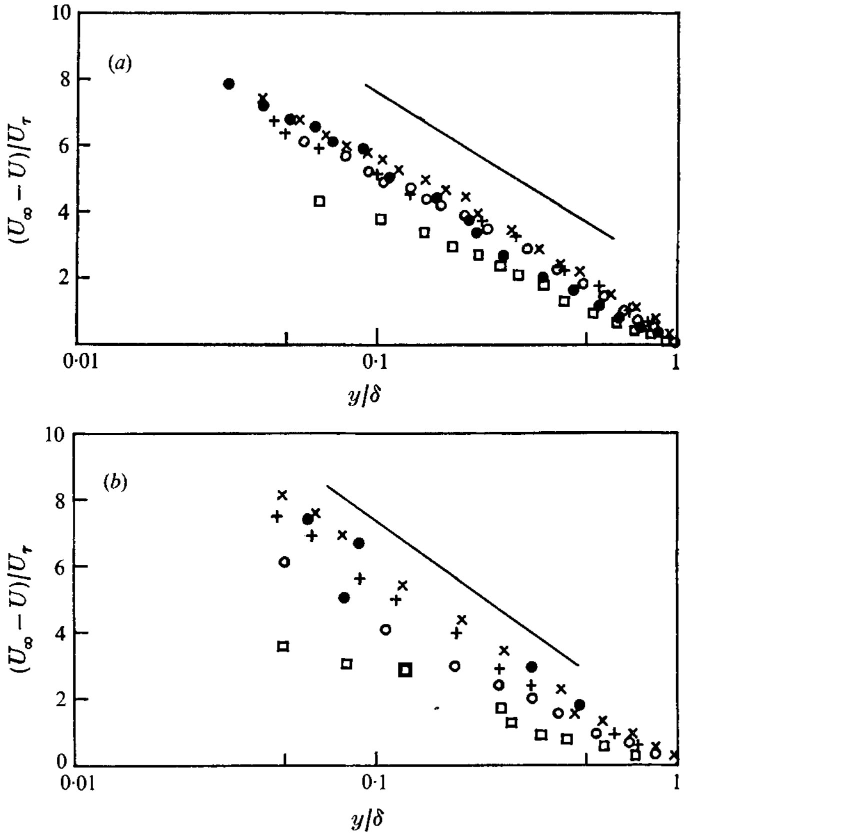


FIGURE 7. Velocity defect law: Rao & Keshavan's measurements. (*a*) $R_a = 3940$. (*b*) $R_a = 825$. ●, $x = 20$ in.; ×, $x = 15·5$ in.; +, $x = 11·5$ in.; ○, $x = 8·5$ in.; □, $x = 3$ in. ——, slope obtained from corresponding law-of-the-wall plot.

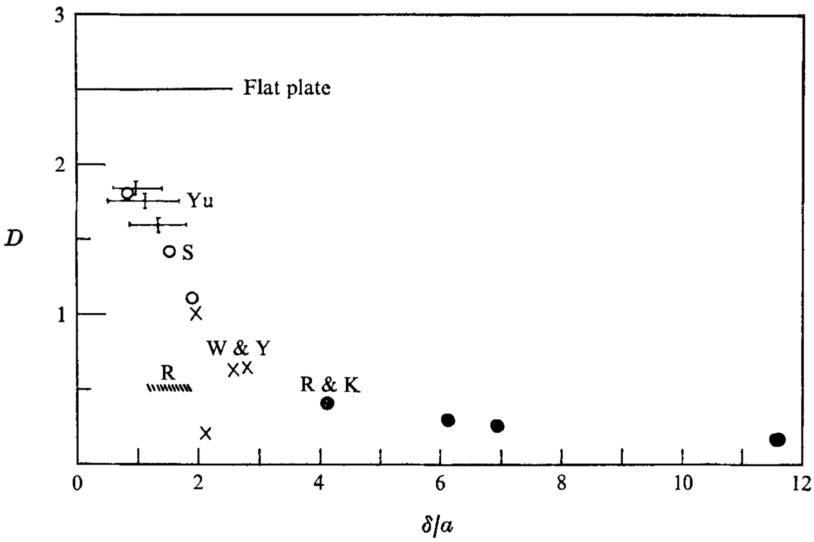


FIGURE 8. Variation of additive term in velocity defect law with respect to δ/a. S, Singh; R, Richmond; R & K, Rao & Keshavan; W & Y, Willmarth & Yang.

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